Math 279 Lecture 3 Notes

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1 Important Stochastic PDEs

1.1 The stochastic heat equation

Last time, we considered the stochastic heat equation

$$u_t = \Delta u + \xi, \qquad x \in \mathbb{R}^d, t \in \mathbb{R}$$

where ξ is space time white noise. We stated that we expect $u \in \mathcal{C}^{-d/2+1}$. In particular, $u \in \mathcal{C}^{1/2}$ in x and $\in \mathcal{C}^{1/4}$ in t when $d = 1$, but when $d > 1$ we don't have a function; it will be a distribution.

Later, we will see how a "subcritical" perturbation can be treated after a "renormalization." To explain this, let us first study the scaling properties of the above stochastic heat equation. Recall that ξ is a 0-mean Gaussian with $\mathbb{E}[\xi(x,t)\xi(y,s)] = \delta_0(x-y,t-s)$. So $\lambda \to \infty$, $\lambda^{d+1}\rho(\lambda x, \lambda^2 t) \to \delta_0(x,t)$. Observe that $\lambda^{d+2}\delta_0(\lambda x, \lambda^2 x) = \delta_0(x,t)$. Hence,

$$\widehat{\xi}(x,t) = \lambda^{(d+2)/2}\xi(\lambda x, \lambda^2 t) \stackrel{d}{=} \xi(x,t).$$

Now we go back to the stochastic heat equation, and if u is a solution, and if $\widehat{u}(x,t) = \lambda^{d/2-1}u(\lambda x, \lambda^2 t)$, then

$$(\widehat{u} - \Delta\widehat{u})(x,t) = \lambda^{d/2+1}(u_t - \Delta u)(\lambda x, \lambda^2 t) = \widehat{\xi} \stackrel{d}{=} \xi.$$

Thus, $\widehat{u}$ is again a solution of the stochastic heat equation. This is compatible with our guess for the Hölder regularity of the solution, namely $u \in \mathcal{C}^{(1-d/2)-}$ in x and $\in \mathcal{C}^{(1/2-d/4)-}$ in t.

1.2 The SHE with multiplicative noise

This PDE looks like

$$Z_t = \Delta Z + \sigma(Z)\xi$$

for a suitable function $\sigma : \mathbb{R} \to \mathbb{R}$.

Two examples that are particularly important are:

1. $\sigma(Z) = \sqrt{Z}$. This example appears in several models in math biology and population dynamics. Imagine you are modeling fish in a lake. Say each fish has an independent exponentially distributed clock that tells you when it dies. When it dies, you replace the fish with a number of descendants.

   Imagine that each particle travels as a Brownian motion, all independent, and after an exponential random time, a particle is replaced with N many particles with $\mathbb{E}[N] = m$. When $m = 1$, we have a critical regime, and as the initial number of particles goes to infinity, we get a measure-valued process known as **super Brownian motion**. When $d = 1$, this measure has a density Z, and Z solves this SHE with multiplicative noise for $\sigma(Z) = \sqrt{Z}$. This is also associated with **Brownian snake**.

2. $\sigma(Z) = Z$. As we will see shortly, this case is related to stochastic growth models.

It turns out that we can make sense of the SHE with multiplicative noise à la Itô. In other words, we write

$$Z(x,t) = \int p(x-y,t)Z(y,0)\,dt + \int_0^t \int p(x-y,t-s)Z(y,s)\underbrace{\xi(y,s)\,dy\,dx}_{W(dy,ds)}$$

when $d = 1$. Note that we still have the Hölder continuous Z multiplied by the distribution ξ. Hairer treated this PDE in 2013.

1.3 The Kardar-Parisi-Zhang equation

We wish to model stochastic growths. Often we have a random interface separating different phases. If the interface can be represented by a graph of a (height) function $h : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, then the Hamilton-Jacobi PDE of the form

$$h_t = H(h_x) \quad (+\Delta u)$$

would be a good model as a first approximation. To capture the roughness of the interface, we may write

$$h_t = H(h_x) + D\Delta h + \lambda\xi.$$

After some manipulation (expanding h about a linear function), we end up with the KPZ equation:

$$h_t = |h_x|^2 + \Delta h + \xi.$$

This is a far more singular PDE than what we have seen before. Note that when $d = 1$, we expect $h \in \mathcal{C}^{1/2-}$, and $h_x \in \mathcal{C}^{-1/2}$.

The main challenge is to make sense of $|h_x|^2$. Indeed, the KPZ equation is "subcritical" only when $d = 1$. To explain this, let h be a solution to this equation, and set $\widehat{h}(x,t) =$

$\lambda^{d/2-1}h(\lambda x, \lambda^2 t)$. Then

$$
\begin{aligned}
(\widehat{h}_t - \Delta \widehat{h})(x,t) &= \underbrace{\lambda^{d/2+1}\xi(\lambda x, \lambda^2 t)}_{\widehat{\xi}(x,t)} + \lambda^{d/2+1}|h_x(\lambda x, \lambda^2 t)|^2 \\
&= \widehat{\xi}(x,t) + \lambda^{1-d/2}|\widehat{h}_x(x,t)|^2.
\end{aligned}
$$

There are a few cases:

1. If $d = 1$, then as $\lambda \to 0$, the nonlinearity disappears. So, locally, the nonlinearity can be ignored!

2. If $d = 2$, this is the *critical regime*. In fact, if we multiply $|h_x|^2$ with a constant of size $\frac{C}{\sqrt{|\log \varepsilon|}}$ (after some smoothing), then we know how to handle the PDE.

3. If $d > 2$, then this is an open problem. We need to replace $\frac{C}{\sqrt{|\log \varepsilon|}}$ with $C\varepsilon^{d/2-1}$.

First observe that if $Z = e^h$ and h solves the KPZ equation, then Z solves the SHE with multiplicative noise. This is called the **Hopf-Cole transform**. This is surprising because the type of singularity we encounter in the KPZ equation is much worse than in the SHE with multiplicative noise. The problem is that the type of solution we had for the SHE with multiplicative noise à la Itô, which means that the usual chain rule must be corrected. Recall that if $\dot{y} = b(x,t) + \sigma dB(t)$, then

$$
d\varphi(y) = \varphi'(y)(b\,dt + \sigma\,dB(t)) + \frac{1}{2}\varphi''(y)\sigma^2\,dt,
$$

where d means the derivative. Recall that $\sum_j (B(t_{j+1}) - B(t_j))^2 \to t$, so $(dB)^2 = dt$. Thus, we get the Itô correction.

Let's go back to Hopf-Cole and do it carefully. To do this carefully, take a smooth kernel χ with $\int \chi = 1$, set $\xi^\varepsilon(x) = \varepsilon^{-d}\xi(x/\varepsilon)$, and put

$$
\xi^\varepsilon(x,t) = \int \chi^\varepsilon(x-y)\xi^\varepsilon(y,t).
$$

Then first solve

$$
Z_t^\varepsilon = Z_{xx}^\varepsilon + \xi^\varepsilon Z^\varepsilon.
$$

Fix x, and treat this equation as a stochastic differential equation in t. Observe that

$$
\begin{aligned}
\mathbb{E}[\xi(x,t)^2] &= \mathbb{E}\left[\left(\int \xi(y,t)\chi^\varepsilon(x-y)\,dy\right)^2\right] \\
&= \mathbb{E}\left[\iint \xi(y,t)\xi(y',t)\chi^\varepsilon(x-y)\chi^\varepsilon(x-y')\,dy\,dy'\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \delta_0(t) \int (\chi^\varepsilon(x-y))^2 \, dy \\
&= \delta_0(t)\varepsilon^{-1} \underbrace{\left(\int \chi^2\right)}_{\bar{c}}.
\end{aligned}
$$

If $h^\varepsilon = \log Z^\varepsilon$, this satisfies

$$h_t^\varepsilon = h_{xx}^\varepsilon + \left[(h_x^\varepsilon)^2 - \frac{1}{2}\bar{c}\varepsilon^{-1}\right] + \xi^\varepsilon.$$

We aimed for the KPZ equation, but letting $h^\varepsilon \to h$, we get that

$$h_t = h_{xx} + (h_x^2 - \infty) + \xi.$$

So we get that this blows up, but we know exactly how.